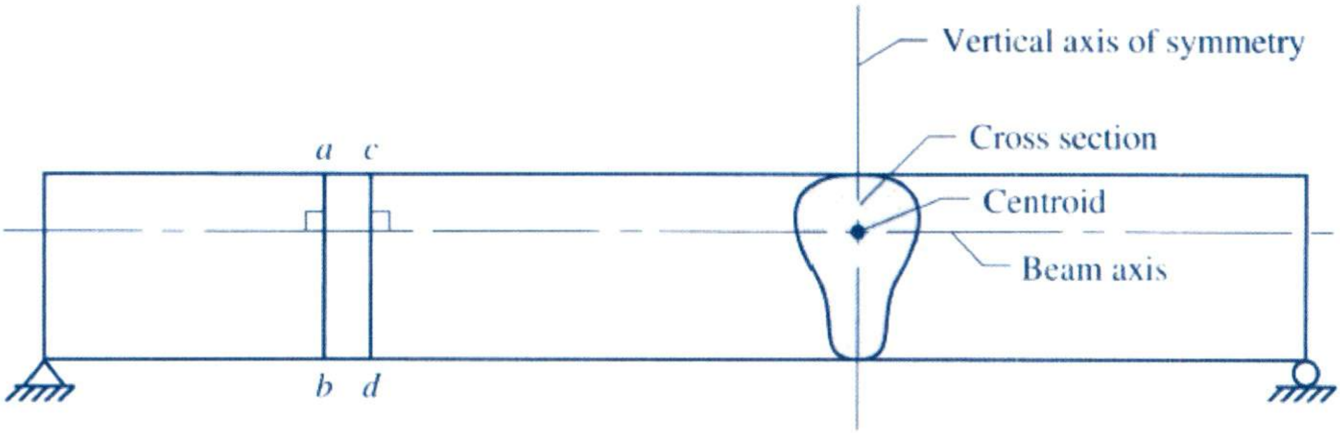
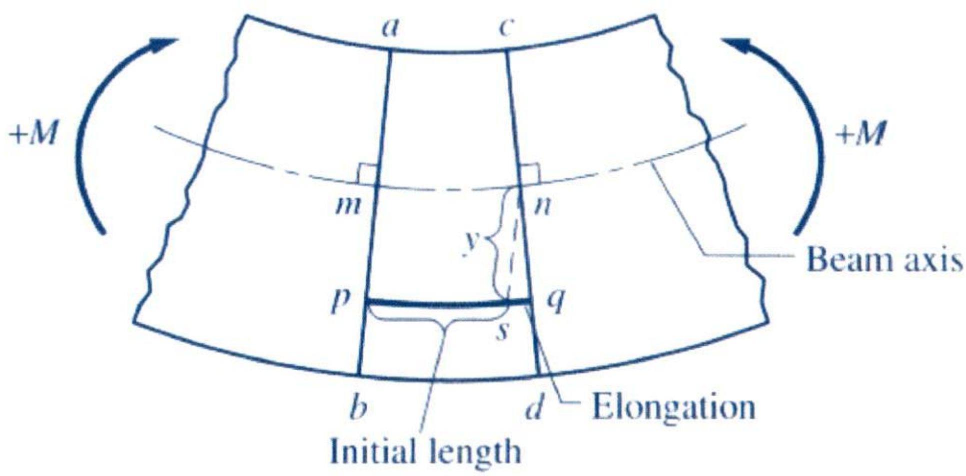
Chapter 14 - Stresses in Beams
Reading: Chapter 14 — Pages 487-510

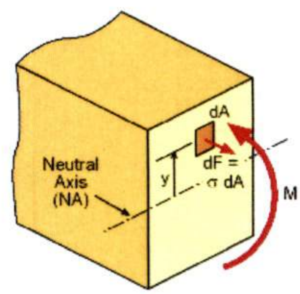
14-1
Introduction

- Normal stresses along the longitudinal direction are caused by bending moments, and shear stresses are caused by shear forces.
- The distribution of normal and shear stresses in a beam and the relationship of these stresses to the internal bending moment and shear force in the beam will be studied.

14-2
Normal Stresses in Beams Due to Bending







Neutral Surface and Neutral Axis

- The fibers mn along the beam axis do not undergo any change of length due to bending, and therefore are not stressed.
- The surface mn is called the *neutral surface.*
- The intersection of the neutral surface with a cross-section is called a *neutral axis.*

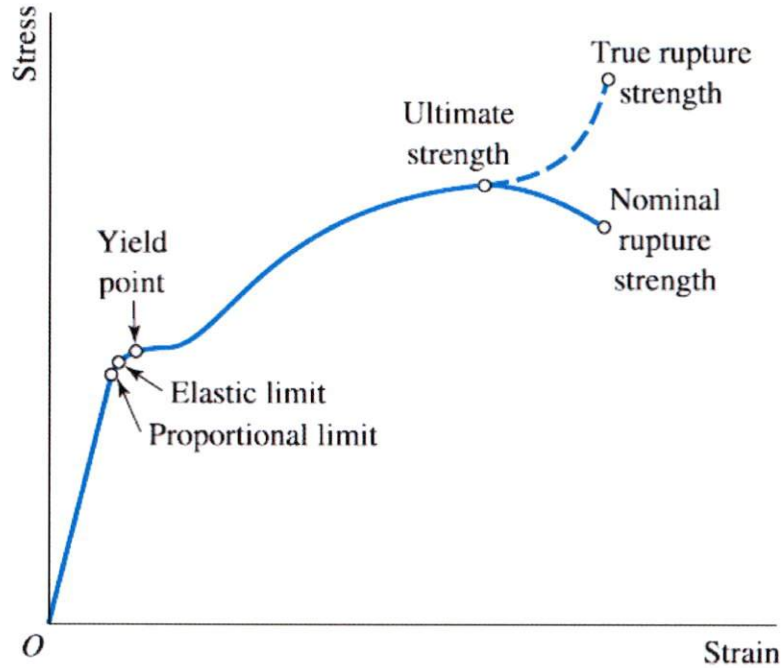
For a beam subjected to pure bending (no axial force), the neutral axis is a horizontal line that passes through the centroid of the cross-sectional area.

Variation of Linear Strains

Linear strains of the longitudinal fibers due to bending vary linearly form zero at the neutral surface to the maximum value at the outer fibers.

Variation of Flexural Stresses

For elastic bending of the beam, Hooke's law applies



Hooke's Law

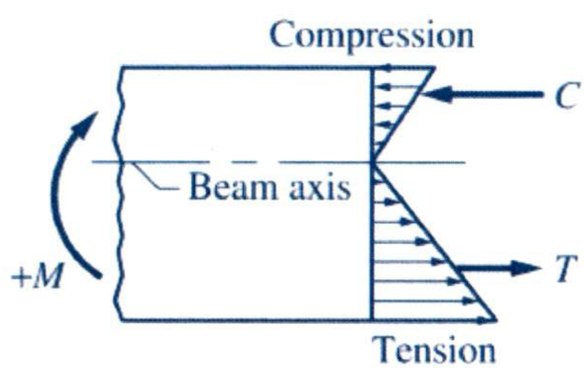
$$\frac{\sigma}{\varepsilon} = E$$

or

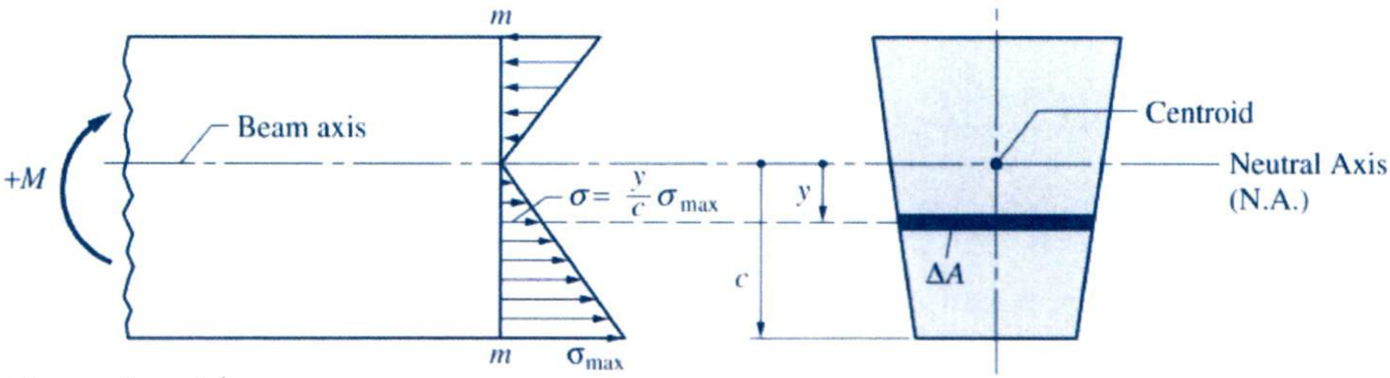
$$\sigma = E\varepsilon$$

where E is the constant of proportionality between stress and strain and is called the *modulus of elasticity*.

- For most materials, the moduli of elasticity in tension and in compression are equal.
- Under these conditions we conclude that the flexural stresses in a beam section vary linearly from zero at the neutral axis to the maximum value at the outer fibers.



The Flexure Formula
Derivation of the Flexure Formula



Normal Stresses

By Similar Triangles,

$$\frac{y}{c} = \frac{\sigma}{\sigma_{MAX}} \Rightarrow \boxed{\sigma = \frac{y}{c}\sigma_{MAX}} \qquad (14\text{-}1)$$

Force on the incremental area ΔA, $P = \sigma \Delta A$

$$\Delta M = (\sigma \Delta A) y$$

$$M = \Sigma \Delta M = \Sigma (\sigma \Delta A) y$$

$$= \Sigma \left(\frac{y}{c}\sigma_{MAX} \Delta A\right) y$$

$$= \frac{\sigma_{MAX}}{c} \Sigma y^2 \Delta A$$

$$= \frac{\sigma_{MAX}}{c} I$$

and $$\boxed{\sigma_{MAX} = \frac{Mc}{I}} \qquad (14\text{-}2)$$

Where,

σ_{MAX} = maximum normal stress due to bending

M = internal resisting moment at the section

c = distance from the neutral axis to the outermost fiber

I = moment of inertia of the cross-sectional area about the neutral axis substitute (14-1) into (14-2)

$$\sigma_{MAX} = \frac{Mc}{I}$$

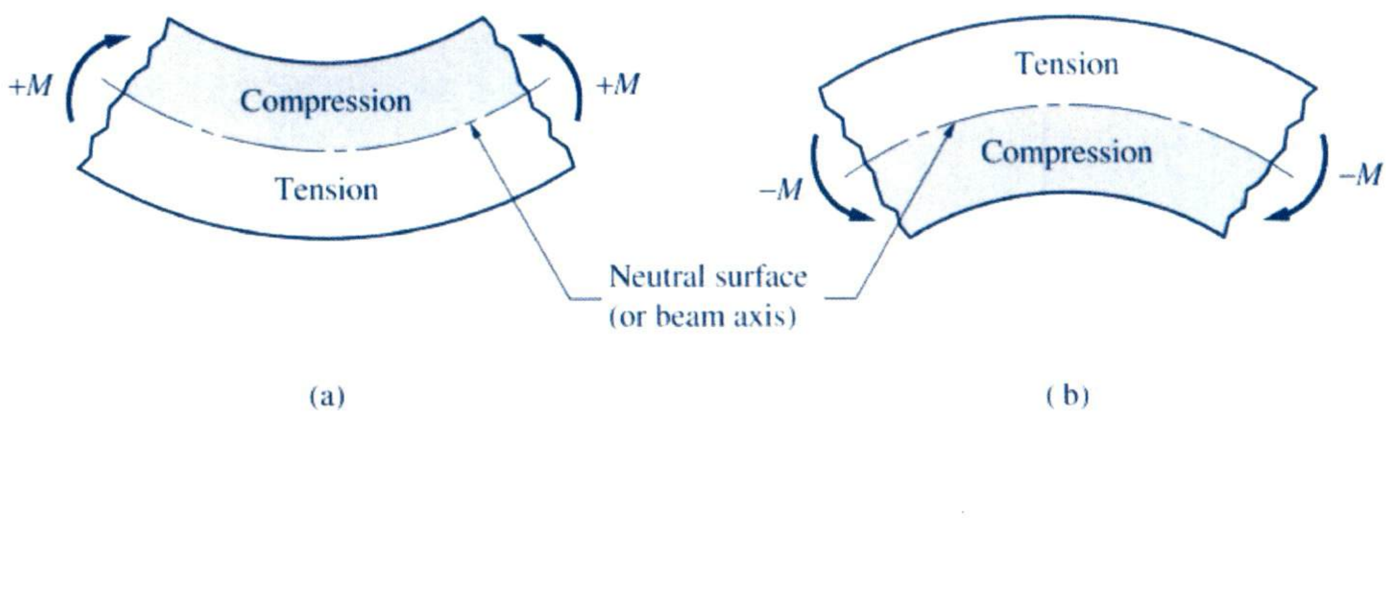
$$\frac{c}{y}\sigma = \frac{Mc}{I}$$

$$\boxed{\sigma = \frac{My}{I}} \qquad (14\text{-}3)$$

where,

σ = the flexural stress at any point in the section

y = the distance from the neutral axis to the point where the flexural stress is desired



(a) (b)

Section Modulus

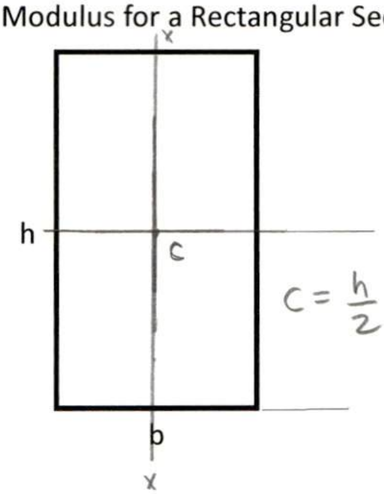
$$S = \frac{I}{c} \implies \frac{c}{I} = \frac{1}{S}$$

From 14-2,

$$\boxed{\sigma_{MAX} = \frac{M}{S}} \qquad (14\text{-}5)$$

most widely used equation

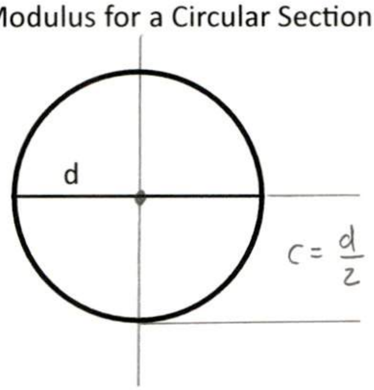
Section Modulus for a Rectangular Section



$$I = \frac{bh^3}{12}$$

$$S = \frac{I}{c} = \frac{\frac{bh^3}{12}}{\frac{h}{2}} = \frac{bh^2}{6} \qquad (14\text{-}6)$$

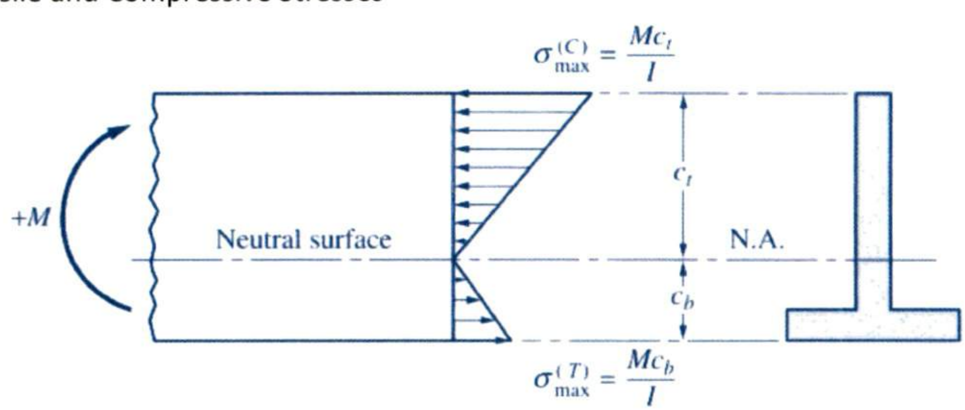
Section Modulus for a Circular Section



$$I = \frac{\pi d^4}{64}$$

$$S = \frac{I}{c} = \frac{\frac{\pi d^4}{64}}{\frac{d}{2}} = \frac{\pi d^3}{32} \qquad (14\text{-}7)$$

Maximum Tensile and Compressive Stresses



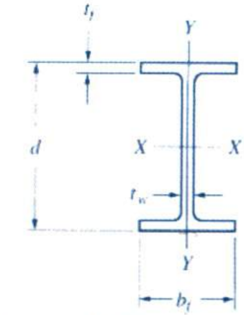


TABLE A–1(a) (Continued) Properties of Selected W Shapes (Wide-Flange Sections): U.S. Customary Units

Designation	Area	Depth	Web Thickness	Flange Width	Flange Thickness	Elastic Properties Axis x–x			Elastic Properties Axis y–y			Plastic Modulus	
(in. × lb/ft)	A (in.^{2})	d (in.)	t_w (in.)	b_f (in.)	t_f (in.)	I (in.^{4})	S (in.^{3})	r (in.)	I (in.^{4})	S (in.^{3})	r (in.)	Z_x (in.^{3})	Z_y (in.^{3})
W14 × 74	21.8	14.17	0.450	10.070	0.785	796	112	6.04	134	26.6	2.48	126	40.6
× 68	20.0	14.04	0.415	10.035	0.720	723	103	6.01	121	24.2	2.46	115	36.9
× 61	17.9	13.89	0.375	9.995	0.645	640	92.2	5.98	107	21.5	2.45	102	32.8
× 53	15.6	13.92	0.370	8.060	0.660	541	77.8	5.89	57.7	14.3	1.92	87.1	22.0
× 43	12.6	13.66	0.305	7.995	0.530	428	62.7	5.82	45.2	11.3	1.89	69.6	17.3
× 38	11.2	14.10	0.310	6.770	0.515	385	54.6	5.87	26.7	7.88	1.55	61.5	12.1
× 34	10.0	13.98	0.285	6.745	0.455	340	48.6	5.83	23.3	6.91	1.53	54.6	10.6
× 30	8.85	13.84	0.270	6.730	0.385	291	42.0	5.73	19.6	5.82	1.49	47.3	8.99
W12 × 87	25.6	12.53	0.515	12.125	0.810	740	118	5.38	241	39.7	3.07	132	60.4
× 65	19.1	12.12	0.390	12.000	0.605	533	87.9	5.28	174	29.1	3.02	96.8	44.1
× 53	15.6	12.06	0.345	9.995	0.575	425	70.6	5.23	95.8	19.2	2.48	77.9	29.1
× 40	11.8	11.94	0.295	8.005	0.515	310	51.9	5.13	44.1	11.0	1.93	57.5	16.8
× 35	10.3	12.50	0.300	6.560	0.520	285	45.6	5.25	24.5	7.47	1.54	51.2	11.5
× 30	8.79	12.34	0.260	6.520	0.440	238	38.6	5.21	20.3	6.24	1.52	43.1	9.56
× 22	6.48	12.31	0.260	4.030	0.425	156	25.4	4.91	4.66	2.31	0.847	29.3	3.66
W10 × 112	32.9	11.36	0.755	10.415	1.250	716	126	4.66	236	45.3	2.68	147	69.2
× 100	29.4	11.10	0.680	10.340	1.120	623	112	4.60	207	40.0	2.65	130	61.0
× 88	25.9	10.84	0.605	10.265	0.990	534	98.5	4.54	179	34.8	2.63	113	53.1
× 77	22.6	10.60	0.530	10.190	0.870	455	85.9	4.49	154	30.1	2.60	97.6	45.9
× 60	17.6	10.22	0.420	10.080	0.680	341	66.7	4.39	116	23.0	2.57	74.6	35.0
× 49	14.4	9.98	0.340	10.000	0.560	272	54.6	4.35	93.4	18.7	2.54	60.4	28.3
× 45	13.3	10.10	0.350	8.020	0.620	248	49.1	4.32	53.4	13.3	2.01	54.9	20.3
× 39	11.5	9.92	0.315	7.985	0.530	209	42.1	4.27	45.0	11.3	1.98	46.8	17.2
× 33	9.71	9.73	0.290	7.960	0.435	170	35.0	4.19	36.6	9.20	1.94	38.8	14.0
× 22	6.49	10.17	0.240	5.750	0.360	118	23.2	4.27	11.4	3.97	1.33	26.0	6.10
W8 × 67	19.7	9.00	0.570	8.280	0.935	272	60.4	3.72	88.6	21.4	2.12	70.2	32.7
× 58	17.1	8.75	0.510	8.220	0.810	228	52.0	3.65	75.1	18.3	2.10	59.8	27.9
× 48	14.1	8.50	0.400	8.110	0.685	184	43.3	3.61	60.9	15.0	2.08	49.0	22.9
× 40	11.7	8.25	0.360	8.070	0.560	146	35.5	3.53	49.1	12.2	2.04	39.8	18.5
× 35	10.3	8.12	0.310	8.020	0.495	127	31.2	3.51	42.6	10.6	2.03	34.7	16.1
× 31	9.13	8.00	0.285	7.995	0.435	110	27.5	3.47	37.1	9.27	2.02	30.4	14.1
× 28	8.25	8.06	0.285	6.535	0.465	98.0	24.3	3.45	21.7	6.63	1.62	27.2	10.1
× 24	7.08	7.93	0.245	6.495	0.400	82.8	20.9	3.42	18.3	5.63	1.61	23.2	8.57
× 21	6.16	8.28	0.250	5.270	0.400	75.3	18.2	3.49	9.77	3.71	1.26	20.4	5.69
× 18	5.26	8.14	0.230	5.250	0.330	61.9	15.2	3.43	7.97	3:04	1.23	17.0	4.66

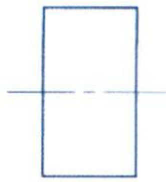


TABLE A–6(a) Properties of Structural Timber: U.S. Customary Units

Nominal Size (in.)	Standard Dressed Size (in.)	Area of Section A (in.^{2})	Moment of Inertia I (in.^{4})	Section Modulus S (in.^{3})	Weight per ft w (lb/ft)
2 × 4	$1\frac{1}{2} \times 3\frac{1}{2}$	5.25	5.36	3.06	1.46
× 6	$\times 5\frac{1}{2}$	8.25	20.8	7.56	2.29
× 8	$\times 7\frac{1}{4}$	10.9	47.6	13.14	3.02
× 10	$\times 9\frac{1}{4}$	13.9	98.9	21.4	3.85
3 × 4	$2\frac{1}{2} \times 3\frac{1}{2}$	8.75	8.93	5.10	2.43
× 6	$\times 5\frac{1}{2}$	13.8	34.7	12.6	3.82
× 8	$\times 7\frac{1}{4}$	18.1	79.4	21.9	5.04
× 10	$\times 9\frac{1}{4}$	23.1	165	35.7	6.42
× 12	$\times 11\frac{1}{4}$	28.1	297	52.7	7.81
4 × 4	$3\frac{1}{2} \times 3\frac{1}{2}$	12.3	12.5	7.15	3.40
× 6	$\times 5\frac{1}{2}$	19.3	48.5	17.6	5.35
× 8	$\times 7\frac{1}{4}$	25.4	111	30.7	7.05
× 10	$\times 9\frac{1}{4}$	32.4	231	49.9	8.93
× 12	$\times 11\frac{1}{4}$	39.4	415	73.8	10.9
× 14	$\times 13\frac{1}{4}$	46.4	678	102	12.9
6 × 6	$5\frac{1}{2} \times 5\frac{1}{2}$	30.3	76.3	27.7	8.40
× 8	$\times 7\frac{1}{2}$	41.3	193	51.6	11.5
× 10	$\times 9\frac{1}{2}$	52.3	393	82.7	14.5
× 12	$\times 11\frac{1}{2}$	63.3	697	121	17.6
× 14	$\times 13\frac{1}{2}$	74.3	1128	167	20.6
× 16	$\times 15\frac{1}{2}$	85.3	1707	220	23.7
× 18	$\times 17\frac{1}{2}$	96.3	2456	281	26.7
8 × 8	$7\frac{1}{2} \times 7\frac{1}{2}$	56.3	264	70.3	15.6
× 10	$\times 9\frac{1}{2}$	71.3	536	113	19.8
× 12	$\times 11\frac{1}{2}$	86.3	951	165	24.0
× 14	$\times 13\frac{1}{2}$	101	1538	228	28.1
× 16	$\times 15\frac{1}{2}$	116	2327	300	32.3
× 18	$\times 17\frac{1}{2}$	131	3350	383	36.5
× 20	$\times 19\frac{1}{2}$	146	4634	475	40.6
10 × 10	$9\frac{1}{2} \times 9\frac{1}{2}$	90.3	679	143	25.1
× 12	$\times 11\frac{1}{2}$	109	1204	209	30.3
× 14	$\times 13\frac{1}{2}$	128	1948	289	35.6
× 16	$\times 15\frac{1}{2}$	147	2948	380	40.9
× 18	$\times 17\frac{1}{2}$	166	4243	485	46.2
× 20	$\times 19\frac{1}{2}$	185	5870	602	51.5
× 22	$\times 21\frac{1}{2}$	204	7868	732	56.7

Note: Properties and weights are for dressed sizes. Weight per unit foot is based on an assumed average weight of 40 lb/ft^3. Moment of inertia and section modulus are about the strong axis.

Example 14-1

A timber section with a nominal 4 in. X 10 in. rectangular section is used on a simple span of 10 ft. The beam supports a uniformly distributed load of 450 lb/ft (which includes the weight of the beam). Determine the maximum flexural stress due to bending.

Solution.

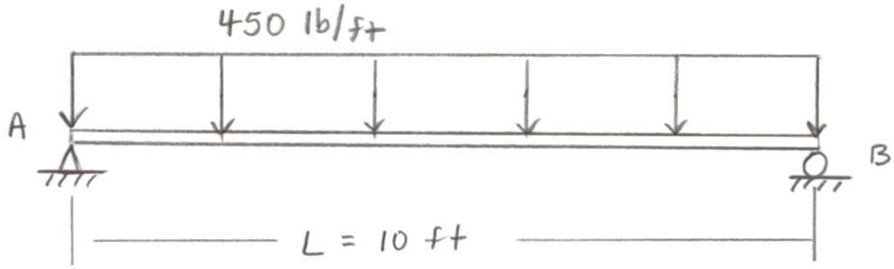


Table 13-1, case 4

$$M_{MAX} = \frac{wL^2}{8} = \frac{450\ \text{lb/ft}\ (10\ \text{ft})^2}{8} = 5625\ \text{lb}\cdot\text{ft}$$

Table A-6(a)

4 in x 10 in - nominal size

3 ½ in x 9 ¼ in. - Standard Dressed size

$I = 231\ \text{in.}^4$

$S = 49.9\ \text{in.}^3$

(EQ 14-2)

$$\sigma_{MAX} = \frac{M_{MAX}\, c}{I} = \frac{5625\ \text{lb}\cdot\text{ft}\left(\frac{12\ \text{in.}}{\text{ft}}\right)\left(\frac{9.25\ \text{in}}{2}\right)}{231\ \text{in.}^4}$$

$$= 1350\ \text{psi}$$

or

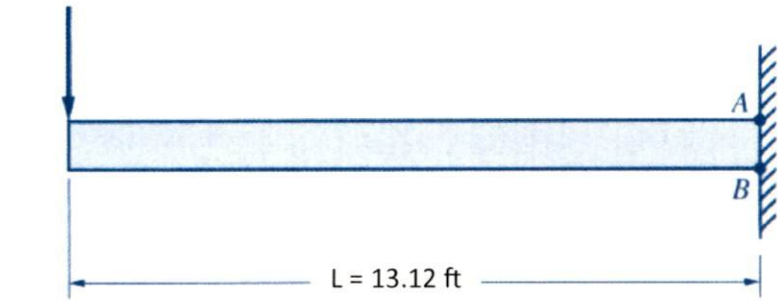
(EQ 14-5)

$$\sigma_{MAX} = \frac{M_{MAX}}{S} = \frac{5625\ \text{lb}\cdot\text{ft}\left(\frac{12\ \text{in.}}{\text{ft}}\right)}{49.9\ \text{in.}^3}$$

$$= 1350\ \text{psi}$$

Example 14-2

A cantilever beam with a 13.12-ft span and a solid circular cross-section of 3.94-in. diameter is subjected to a concentrated load P = 450 lb applied at the free end, as shown in Fig. E14-2A. Determine the maximum flexural stress in the beam caused by the load.



Solution.

Table 13-1, case 5

$$M_{MAX} = -Pa = -PL = -450\text{ lb }(13.12\text{ ft}) = -5904\text{ lb.ft}$$

Section modulus (circular Section)

$$S = \frac{\pi d^3}{32} = \frac{\pi (3.94\text{ in})^3}{32} = 6.0\text{ in.}^3$$

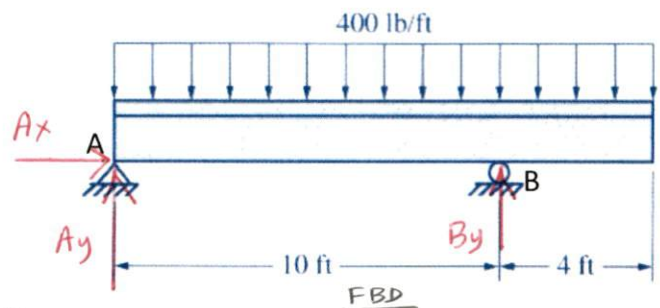
Maximum Flexural Stress

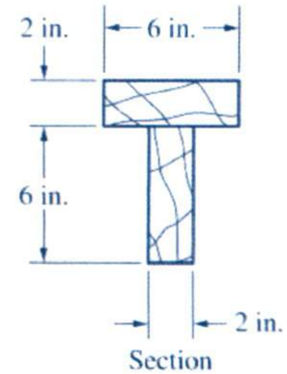
$$\sigma_{MAX} = \frac{M_{MAX}}{S} = \frac{5904\text{ lb.ft}\left(\frac{12\text{ in.}}{\text{ft}}\right)}{6.0\text{ in.}^3}$$

$$= 11,808\text{ psi}$$

Example 14-3

The overhanging beam in Fig. E14-3(1) is built up with two full-size timber planks, 2 in. x 6 in., glued together to form a T-section, as shown in Fig. E14-3(2). The beam is subjected to a uniform load of 400 lb/ft, which includes the weight of the beam. Determine the maximum tensile and compressive flexural stresses in the beam.





Solution.

Equilibrium Equations

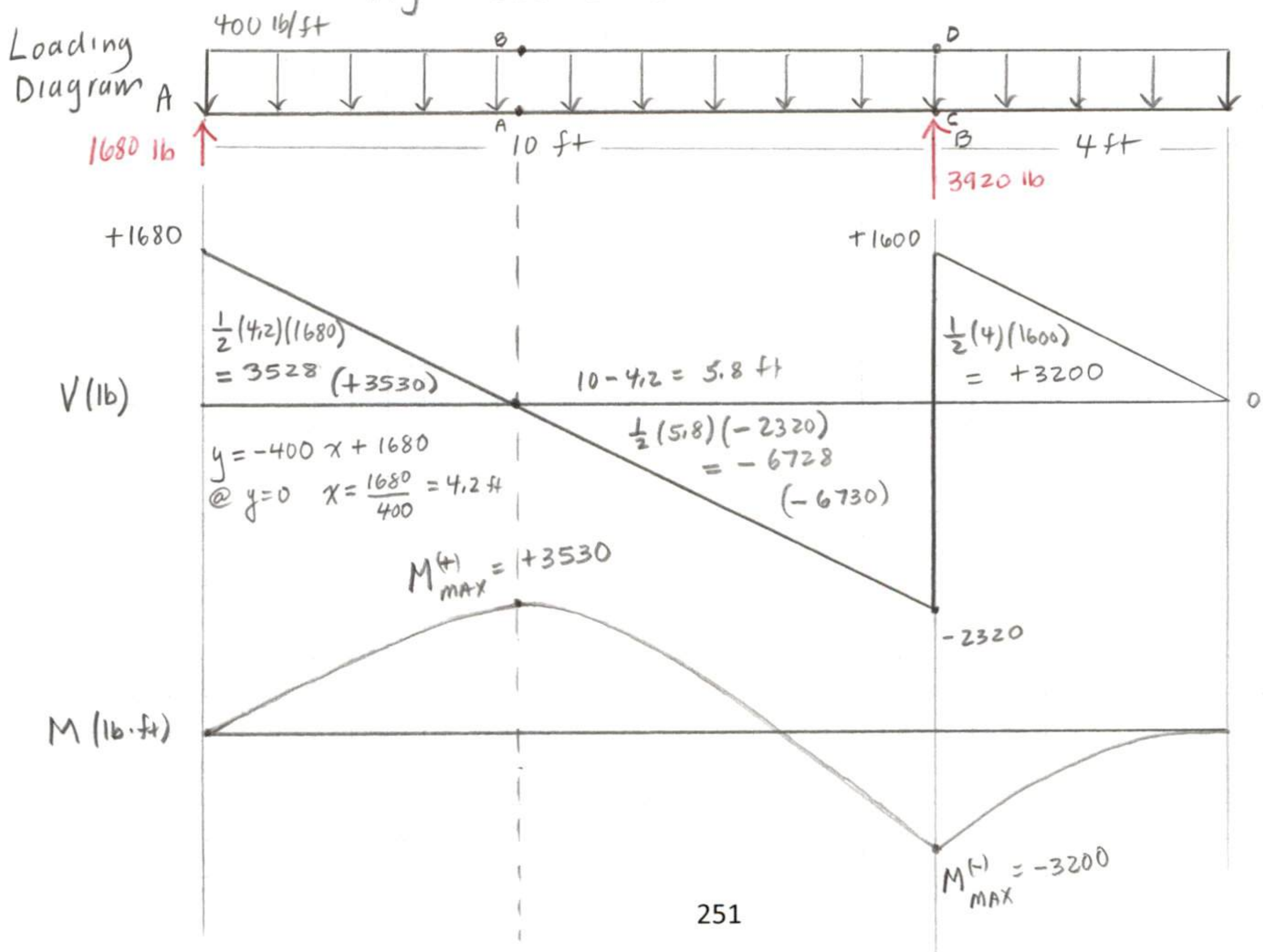
$[\Sigma F_x = 0] \quad A_x = 0$

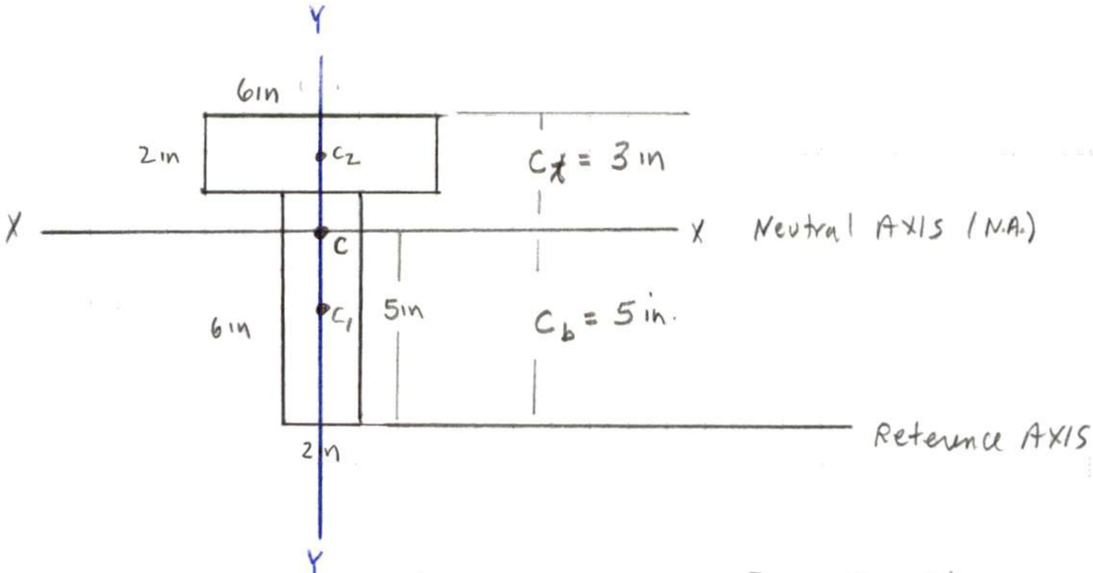
$+\circlearrowleft [\Sigma M_A = 0] \quad -400\,\frac{lb}{ft}(14\,ft)(7\,ft) + B_y(10\,ft) = 0$

$B_y = \frac{39200\ lb\cdot ft}{10\ ft} = 3920\ lb \uparrow$

$[\Sigma F_y = 0] \quad A_y - 5600\ lb + B_y = 0$

$A_y = 5600\ lb - 3920\ lb = 1680\ lb \uparrow$





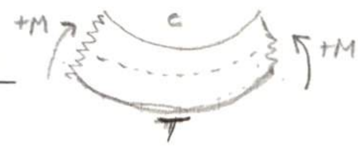
Shape	Area (in.^{2})	y (in)	Ay (in.^{3})	$\bar{y}-y$	$A(\bar{y}-y)^2$	I (in.^{4})
A1	2×6 = 12	3	36	2	48	$\frac{(2)6^3}{12}$ = 36
A2	6×2 = 12	7	84	−2	48	$6(2^3)/12$ = 4
Σ	24		120		96	40

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{120}{24} = 5 \text{ in.}$$

moment of Inertia for the Section

$$I_{NA} = \Sigma\left[I + A(\bar{y}-y)^2\right] = 40 \text{ in.}^4 + 96 \text{ in}^4 = 136 \text{ in}^4.$$

Flexural Stress at $M^{(+)}_{MAX}$



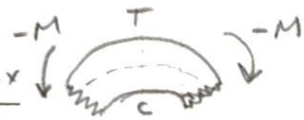
Maximum Tensile Stress occurs at A (bottom fibers)

$$\sigma_A = \frac{M^{(+)}_{max} C_b}{I} = \frac{3530 \text{ lb}\cdot\text{ft}\left(\frac{12 \text{ in}}{\text{ft}}\right)(5 \text{ in})}{136 \text{ in.}^4} = 1560 \text{ psi (T)}$$

MAX Compressive Stress occurs at B (top fibers)

$$\sigma_B = \frac{M^{(+)}_{max} C_t}{I} = \frac{3530 \text{ lb}\cdot\text{ft}\left(\frac{12 \text{ in}}{\text{ft}}\right)(3 \text{ in})}{136 \text{ in.}^4} = 934 \text{ psi (C)}$$

Flexural Stress at $M^{(-)}_{MAX}$



MAX Tensile Stress occurs at D (Top fibers)

$$\sigma_D = \frac{M^{(-)}_{max} C_t}{I} = \frac{3200 \text{ lb}\cdot\text{ft}\left(\frac{12 \text{ in}}{\text{ft}}\right)(3 \text{ in})}{136 \text{ in}^4} = 953 \text{ psi (T)}$$

MAX Compressive Stress occurs at C (bottom fibers)

$$\sigma_C = \frac{M^{(-)}_{max} C_b}{I} = \frac{3200 \text{ lb}\cdot\text{ft}\left(\frac{12 \text{ in}}{\text{ft}}\right)(5 \text{ in})}{136 \text{ in}^4} = 1410 \text{ psi (C)}$$

MAX Tensile Stress

$$\sigma^{(T)}_{MAX} = \sigma_A = \underline{\underline{1560 \text{ psi}}}$$

MAX compressive Stress

$$\sigma^{(C)}_{MAX} = \sigma_C = \underline{\underline{1410 \text{ psi}}}$$

14-4

Allowable Moment

Solving EQ 14-2 for the moment M and using the allowable flexural stress σ_{allow} for σ_{max} we get the formula for computing the allowable moment of a beam:

$$\sigma_{MAX} = \frac{Mc}{I}$$

$$\boxed{M_{allow} = \frac{I\,\sigma_{allow}}{c}} \qquad (14\text{-}8)$$

where

M_{allow} = the allowable moment of a beam

I = the moment of inertia of the cross-sectional area about the neutral axis

σ_{allow} = the allowable flexural stress of the beam

c = the distance from the neutral axis to the outermost fiber

also, since $S = \frac{I}{c}$ (section modulus)

$$\boxed{M_{allow} = S\,\sigma_{allow}} \qquad (14\text{-}9)$$

Allowable moment is mainly computed for the purpose of computing the Allowable Load that can be applied safely to the beam without causing over-stress of the beam.

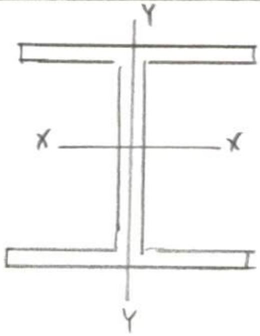
Example 14-4

Determine the allowable uniform load that a structural steel W14 x 38 beam can support over a simple span of 12 ft without exceeding an allowable flexural stress of 24 ksi.

$\sigma_{allow} = 24 \text{ ksi}$

Solution.

From Table A-1(a) (Pg 764 Textbook)



W 14×38

$S_x = 54.6 \text{ in}^3$

Beam weight

$w = 38 \text{ lb/ft}$

$$M_{allow} = S\,\sigma_{allow}$$

$$= 54.6 \text{ in}^3 \,(24 \text{ ksi})$$

$$= 54.6 \text{ in}^3 \left(24 \frac{\text{kip}}{\text{in}^2}\right)$$

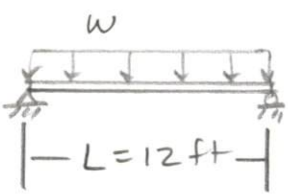
$$= 1310 \text{ kip}\cdot\text{in} \times \frac{1 \text{ ft}}{12 \text{ in}}$$

$$= 109.2 \text{ kip}\cdot\text{ft}$$

Simple Beam Span

uniform Load

Table 13-1, Case 4



$$M_{MAX} = \frac{wL^2}{8}$$

$$w = \frac{8\,(M_{allow})}{L^2} = \frac{8(109.2 \text{ kip}\cdot\text{ft})}{(12 \text{ ft})^2}$$

$$= 6.067 \text{ kip/ft}$$

$$= 6067 \frac{\text{lb}}{\text{ft}}$$

$$w_{allow} = \text{Load}_{allow} - \text{Beam Weight}$$

$$= 6067 \frac{\text{lb}}{\text{ft}} - 38 \frac{\text{lb}}{\text{ft}} = \underline{\underline{6030. \text{ lb/ft}}}$$